



Overview of Generalized Disjunctive Programming

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Motivation



- **EWO Problems involve Discrete/Continuous Optimization**

- ◆ Linear/Nonlinear models
- ◆ 0-1 and continuous decisions

- **Optimization Models**

- ◆ Mixed-Integer Linear Programming (**MILP**)
- ◆ Mixed-Integer Nonlinear Programming (**MINLP**)

Alternative approach:

- ◆ Logic-based: **Generalized Disjunctive Programming (GDP)**

- **Outline**

- ◆ Modeling with GDP (**LOGMIP**)
- ◆ Solution methods (relaxations)
- ◆ Example **supply chain with contracts for purchases**
- ◆ New frontiers:
 - Improved relaxations (basic steps) (**Strip packing problem**)
 - Nonconvex GDPs

MI(N)LP

- **Mixed-Integer Linear/Nonlinear Programming**

$$\min Z = f(x, y)$$

Objective Function

$$s.t. \quad h(x, y) = 0$$

Equality Constraints

$$g(x, y) \leq 0$$

Inequality Constraints

$$x \in R^n, \quad y \in \{0,1\}^m$$

MILP: $f(x,y)$ and $g(x,y)$ **linear** in x and y

MINLP: $f(x,y)$ and $g(x,y)$ – commonly assumed to be **convex and bounded**
often not true in practice => nonconvex

$f(x,y)$ and $g(x,y)$ commonly **linear** in y *often true in practice*



Mixed-integer Nonlinear Programming



Codes MILP:

Commercial : CPLEX, XPRESS, (**GUROBI ?**) XA, OSL

Open Source (COIN-OR) : CBC, SYMPHONY

Codes MINLP:

SBB *GAMS simple B&B*

MINLP-BB (AMPL) *Fletcher and Leyffer (1999)*

Bonmin (COIN-OR) *Bonami et al (2006)*

FilMINT *Linderoth and Leyffer (2006)*

DICOPT (GAMS) *Viswanathan and Grossman (1990)*

AOA (AIMSS)

α -ECP *Westerlund and Petersson (1996)*

MINOPT *Schweiger and Floudas (1998)*

BARON *Sahinidis et al. (1998)*

Global solvers

Couenne (COIN-OR) *Belotti et al. (2008)*



Generalized Disjunctive Programming



Motivation

1. Facilitate modeling of discrete/continuous optimization problems through use algebraic constraints and *symbolic logic expressions*
2. Improve combinatorial search effort
3. Improve handling nonlinearities



Basic References

Definition of GDP

Raman, R. and I.E. Grossmann, "Modeling and Computational Techniques for Logic Based Integer Programming," *Computers and Chemical Engineering*, 18, 563 (1994).

Logic-based Outer-Approximation

Turkay, M. and I.E. Grossmann, "Logic-Based MINLP Algorithms For the Optimal Synthesis of Process Networks," *Computers and Chemical Engineering*, 20, 959-978 (1996).

Convex hull nonlinear disjunctions

Lee, S. and I.E. Grossmann, "New Algorithms for Nonlinear Generalized Disjunctive Programming," *Computers and Chemical Engineering*, 24, pp.2125-2141 (2000).

Established relation with DP

Sawaya, N. and I.E. Grossmann, "Reformulations, Relaxations and Cutting Planes for Linear Generalized Disjunctive Programming," submitted for publication *Math. Prog.* (2008).

LOGMIP

Vecchietti, A. and I.E. Grossmann, "LOGMIP: A Disjunctive 0-1 Nonlinear Optimizer for Process Systems Models," *Computers and Chemical Engineering* 23, 555-565 (1999).



Background reading

Balas E., “Disjunctive Programming and a Hierarchy of Relaxations for Discrete Continuous Optimization Problems”, *SIAM J. Alg. Disc. Meth.*, Vol. 6, No. 3, 1985.

Hooker J., “Logic-based methods for optimization: combining optimization and constraint satisfaction”, John Wiley & Sons, 2000.

Jeroslow R.G., “Logic-Based Decision Support: Mixed-Integer Model Formulation”, *Annals of Discrete Mathematics*, 40, North Holland (Amsterdam), 1989.

“Typical” disjunction for scheduling

Product A and Product B must be processed on same line

Sequencing decision:

A before B **OR** B before A

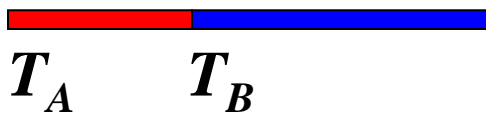
Let T_A , T_B be starting times (*variables*)

Let p_A , p_B be processing times (*parameters*)

Disjunction

$$\left[T_A + p_A \leq T_B \right] \vee \left[T_B + p_B \leq T_A \right]$$

A before B **OR** **B before A**



Generalized Disjunctive Programming (GDP)

Quantitative/Symbolic Optimization Model

- Raman and Grossmann (1994) (*Extension Balas, 1979*)

	$\min Z = \sum_k c_k + f(x)$	Objective Function
	$s.t. \quad r(x) \leq 0$	Common Constraints
OR operator →	$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c_k = \gamma_{jk} \end{bmatrix} \quad k \in K$	Disjunction Constraints
	$\Omega(Y) = true$	Fixed Charges
	$x \in R^n, c_k \in R^1$	Logic Propositions
	$Y_{jk} \in \{ true, false \}$	Continuous Variables
		Boolean Variables

Small GDP Problem

$$\min Z = c + 2x_1 + x_2$$

s.t.

$$\left[\begin{array}{c} Y_1 \\ -x_1 + x_2 + 2 \leq 0 \\ c = 5 \end{array} \right] \vee \left[\begin{array}{c} Y_2 \\ 2 - x_2 \leq 0 \\ c = 7 \end{array} \right]$$

$$\left[\begin{array}{c} Y_3 \\ x_1 - x_2 \leq 1 \end{array} \right] \vee \left[\begin{array}{c} \neg Y_3 \\ x_1 = 0 \end{array} \right]$$

$$Y_1 \wedge \neg Y_2 \Rightarrow \neg Y_3$$

$$\neg(Y_2 \wedge Y_3)$$

$$0 \leq x_1 \leq 5, 0 \leq x_2 \leq 5, c \geq 0$$

$$Y_j \in \{true, false\}, j = 1, 2, 3.$$

Input file LOGMIP (GAMS)

```
SET I /1*3/;
SET J /1*2/;
BINARY VARIABLES Y(I);
POSITIVE VARIABLES X(J), C;
VARIABLE Z;
EQUATIONS EQUAT1, EQUAT2, EQUAT3, EQUAT4,
    EQUAT5, EQUAT6, DUMMY, OBJECTIVE;
```

```
EQUAT1.. X('2')- X('1') + 2 =L= 0;
EQUAT2.. C =E= 5;
EQUAT3.. 2 - X('2') =L= 0;
EQUAT4.. C =E= 7;
EQUAT5.. X('1')-X('2') =L= 1;
EQUAT6.. X('1') =E= 0;
DUMMY.. SUM(I, Y(I)) =G= 0;
OBJECTIVE.. Z =E= C + 2*X('1') + X('2');
X.UP(J)=20;
C.UP=7;
```

```
$ONTEXT BEGIN LOGMIP
DISJUNCTION D1, D2;
D1 IS
IF Y('1') THEN
    EQUAT1;
    EQUAT2;
ELSIF Y('2') THEN
    EQUAT3;
    EQUAT4;
ENDIF;

D2 IS
IF Y('3') THEN
    EQUAT5;
ELSE
    EQUAT6;
ENDIF;
Y('1') and not Y('2') -> not Y('3');
Y('2') -> not Y('3') ;
Y('3') -> not Y('2') ;
$OFFTEXT END LOGMIP
OPTION MIP=LOGMIPM;
MODEL PEQUE2 /ALL/;
SOLVE PEQUE2 USING MIP MINIMIZING Z;
```

**Big-M
Formulation**



LogMIP



Aldo Vecchietti, INGAR

Part of GAMS Modeling System

-Disjunctions specified with **IF Then ELSE** statements

DISJUNCTION D1(I,K,J);

D1(I,K,J)

with (L(I,K,J)) IS

IF Y(I,K,J) THEN

NOCLASH1(I,K,J);

ELSE

NOCLASH2(I,K,J);

ENDIF;

-Logic can be specified in symbolic form (\Rightarrow , OR, AND, NOT)

or special operators (ATMOST, ATLEAST, EXACTLY)

-Linear case: **MILP reformulation big-M, convex hull**

-Nonlinear: **Logic-based OA**

<http://www.ceride.gov.ar/logmip/>

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Generalized Disjunctive Programming (GDP)

OR operator \longrightarrow

$$\begin{aligned} \min Z &= \sum_k c_k + f(x) \\ \text{s.t.} \quad &r(x) \leq 0 \\ &\bigvee_{j \in J_k} \left[\begin{array}{l} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c_k = \gamma_{jk} \end{array} \right] \quad k \in K \\ &\Omega(Y) = \text{true} \\ &x \in R^n, \quad c_k \in R^1 \\ &Y_{jk} \in \{ \text{true}, \text{false} \} \end{aligned}$$

Objective Function

Common Constraints

Disjunction

Constraints

Fixed Charges

Logic Propositions

Continuous Variables

Boolean Variables

Solution methods: Relaxation?

Systematic Procedure to Derive Linear Inequalities for Logic Propositions

Goal is to Convert Logical Expression into

Conjunctive Normal Form (CNF): $Q_1 \wedge Q_2 \wedge \dots \wedge Q_s$

where *clause* $Q_i: P_1 \vee P_2 \vee \dots \vee P_r$ and P_i is a *literal* $\Rightarrow \sum_{i \in \text{Nonneg}} y_i + \sum_{i \in \text{Neg}} (1 - y_i) \geq 1$

Linear inequality

Steps to obtain CNF (Clocksin, Melish, 1981)

1. Replace implication by disjunction

$$P1 \Rightarrow P2 \Leftrightarrow \neg P1 \vee P2 \quad \neg \text{negation}$$

2. Move negation inward applying De Morgan's theorem

$$\neg (P1 \wedge P2) \Leftrightarrow \neg P1 \vee \neg P2$$

$$\neg (P1 \vee P2) \Leftrightarrow \neg P1 \wedge \neg P2$$

3. Recursively distribute or over and

$$(P1 \wedge P2) \vee P3 \Leftrightarrow (P1 \vee P3) \wedge (P2 \vee P3)$$

Example

If prod A or prod B implies reactor 3 or reactor 4

$$P_1 \quad P_2 \quad P_3 \quad P_4$$

$$P_1 \vee P_2 \Rightarrow P_3 \vee P_4$$

1. Remove implication

$$\neg(P_1 \vee P_2) \vee (P_3 \vee P_4)$$

2. Apply De Morgan's

$$(\neg P_1 \wedge \neg P_2) \vee (P_3 \vee P_4)$$

3. Distribute OR over AND

$$(\underbrace{\neg P_1 \vee P_3 \vee P_4}_{\text{clause 1}}) \wedge (\underbrace{\neg P_2 \vee P_3 \vee P_4}_{\text{clause 2}}) \Rightarrow \text{CNF !}$$

from clause 1

$$1 - y_1 + y_3 + y_4 \geq 1$$

from clause 2

$$1 - y_2 + y_3 + y_4 \geq 1$$

$$y_3 + y_4 \geq y_1$$

$$y_3 + y_4 \geq y_2$$

Big-M MI(N)LP (BM)

- **MINLP reformulation of GDP**

$$\begin{aligned}
 \min Z &= \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x) \\
 \text{s.t.} \quad &r(x) \leq 0 \\
 &g_{jk}(x) \leq M_{jk} (1 - \lambda_{jk}) \quad j \in J_k, k \in K \\
 &\sum_{j \in J_k} \lambda_{jk} = 1, \quad k \in K \\
 &A\lambda \leq a \\
 &x \geq 0, \lambda_{jk} \in \{0, 1\}
 \end{aligned}$$

Big-M Parameter \rightarrow (points to M_{jk})
 Logic constraints \leftarrow (points to $A\lambda \leq a$)

LP/NLP Relaxation $0 \leq \lambda_{jk} \leq 1$

Convex Hull Formulation (CH)

- Consider **Disjunction** $k \in K$

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c = \gamma_{jk} \end{bmatrix}$$

◆ **Theorem:** Convex Hull of Disjunction k (Lee, Grossmann, 2000)

- **Disaggregated variables** v^j

$$\{(x, c) \mid x = \sum_{j \in J_k} v^{jk}, \quad c_k = \sum_{j \in J_k} \lambda_{jk} \gamma_{jk},$$

$$0 \leq v^{jk} \leq \lambda_{jk} U_{jk}, \quad j \in J_k$$

\Rightarrow **Convex Constraints**

$$\sum_{j \in J_k} \lambda_{jk} = 1, \quad 0 < \lambda_{jk} \leq 1,$$

$$\lambda_{jk} g_{jk}(v^{jk} / \lambda_{jk}) \leq 0, \quad j \in J_k \}$$

Convex for $g(x)$ convex

- λ_j - weights for linear combination

Remark I

For linear disjunctions

$$\bigvee_{j \in J_k} \left[A_{jk} x \leq b_{jk} \right]$$

Convex-hull set $g_{jk}(x) = A_{jk}x - b_{jk}$

$$x = \sum_{j \in J_k} v_{jk}$$

$$A_{jk} v_{jk} \leq b_{jk} \lambda_{jk} \quad j \in J_k \quad \text{Balas (1985)}$$


$$\sum_{j \in J_k} \lambda_{jk} = 1$$

$$0 \leq \lambda_{jk} \leq 1 \quad j \in J_k$$

Remark II

For special cases disaggregated variables can be eliminated

Example $\left[\begin{array}{c} Y \\ L \leq x \leq U \end{array} \right] \vee \left[\begin{array}{c} \neg Y \\ x = 0 \end{array} \right]$



Convex hull formulation

$$x = v^1 + v^2$$

$$L\lambda \leq v^1 \leq U\lambda$$

$$v^2 = 0 * (1 - \lambda)$$

$$\lambda = 0, 1$$

Since $v^2 = 0 \Rightarrow$

$$\begin{array}{l} L\lambda \leq x \leq U\lambda \\ \lambda = 0, 1 \end{array}$$

w-MIP representable (Raman, G, 1994)

See also John Hooker EWO Seminar

Tour Modeling Techniques

Remark III

Nonlinear disjunctions

How to implement $\lambda_{jk} g_{jk}(v_{jk} / \lambda_{jk}) \leq 0$ for λ_{jk} zero ?

Replace $\lambda_{jk} g_{jk}(v_{jk} / \lambda_{jk}) \leq 0$ where $0 \leq v_{jk} \leq U \lambda_{jk}$ by:

$$((1 - \varepsilon)\lambda_{jk} + \varepsilon)(g_{jk}(v_{jk} / ((1 - \varepsilon)\lambda_{jk} + \varepsilon))) - \varepsilon g_{jk}(0)(1 - \lambda_{jk}) \leq 0$$

Furman, Sawaya & Grossmann (2009)

- Exact approximation of the original constraints as $\varepsilon \rightarrow 0$.
- The constraints are exact at $\lambda_{jk} = 0$ and at $\lambda_{jk} = 1$ regardless of value of ε .
 if $\lambda_{jk} = 0$, $\Rightarrow (\varepsilon)(g_{jk}(0)) - \varepsilon g_{jk}(0) = 0 \leq 0$
 if $\lambda_{jk} = 1$, $\Rightarrow ((1)(g_{jk}(v_{jk} / (1))) - \varepsilon g_{jk}(0)(0)) = (1)g_{jk}(v_{jk} / (1)) \leq 0$
- The LHS of the new constraint is **convex**.

MI(N)LP Reformulation (CH)

$$\min Z = \sum_{k \in K} \sum_{j \in J_k} \gamma_{jk} \lambda_{jk} + f(x)$$

$$s.t. \quad r(x) \leq 0$$

$$x = \sum_{j \in J_k} v^{jk}, k \in K$$

$$0 \leq v^{jk} \leq \lambda_{jk} U_{jk}, j \in J_k, k \in K$$

$$\sum_{j \in J_k} \lambda_{jk} = 1, k \in K$$

$$\lambda_{jk} g_{jk}(v^{jk} / \lambda_{jk}) \leq 0, j \in J_k, k \in K$$

$$A\lambda \leq a$$

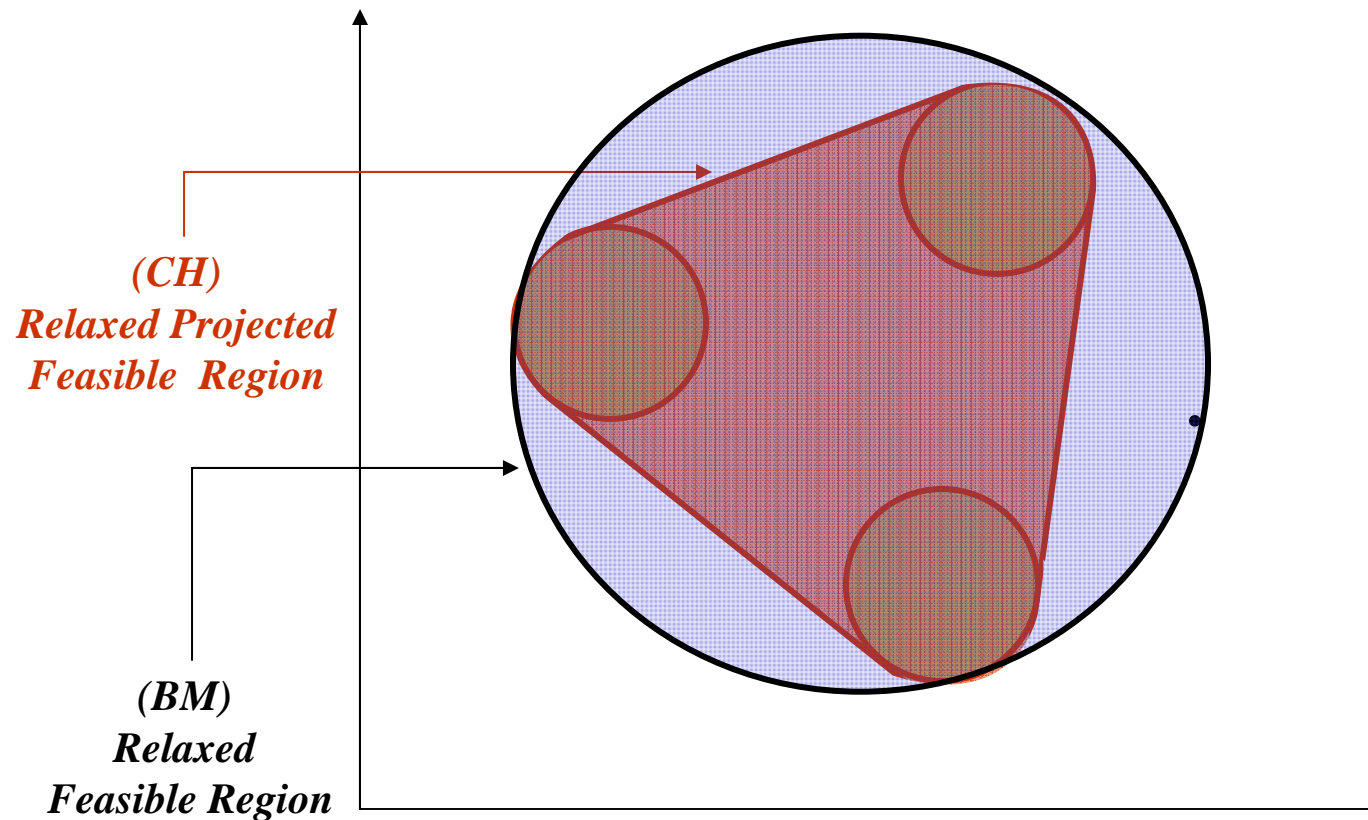
$$x, v^{jk} \geq 0, \lambda_{jk} = 0, 1 \quad j \in J_k, k \in K$$

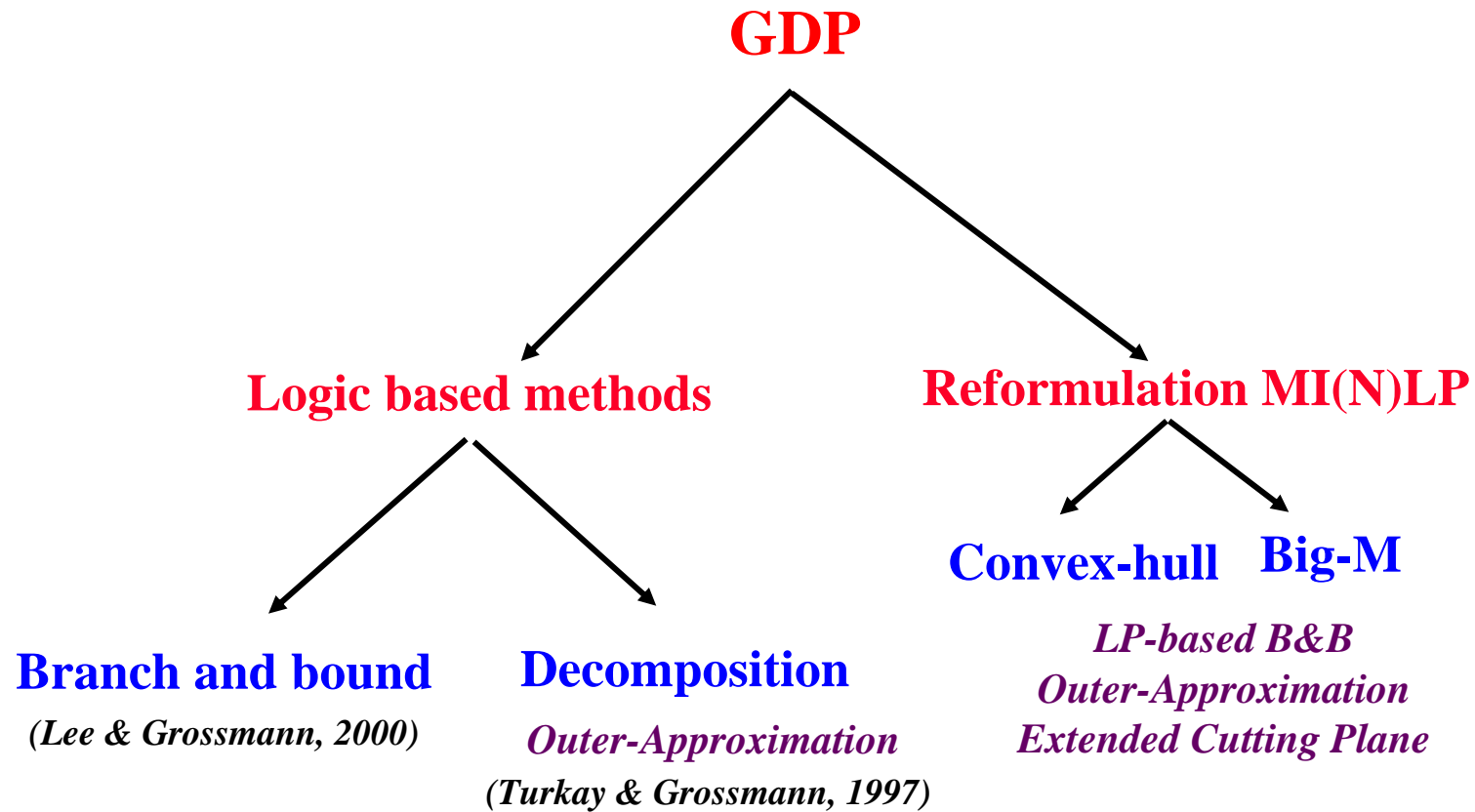
LP/NLP Relaxation $0 \leq \lambda_{jk} \leq 1$

Convex Hull vs Big-M

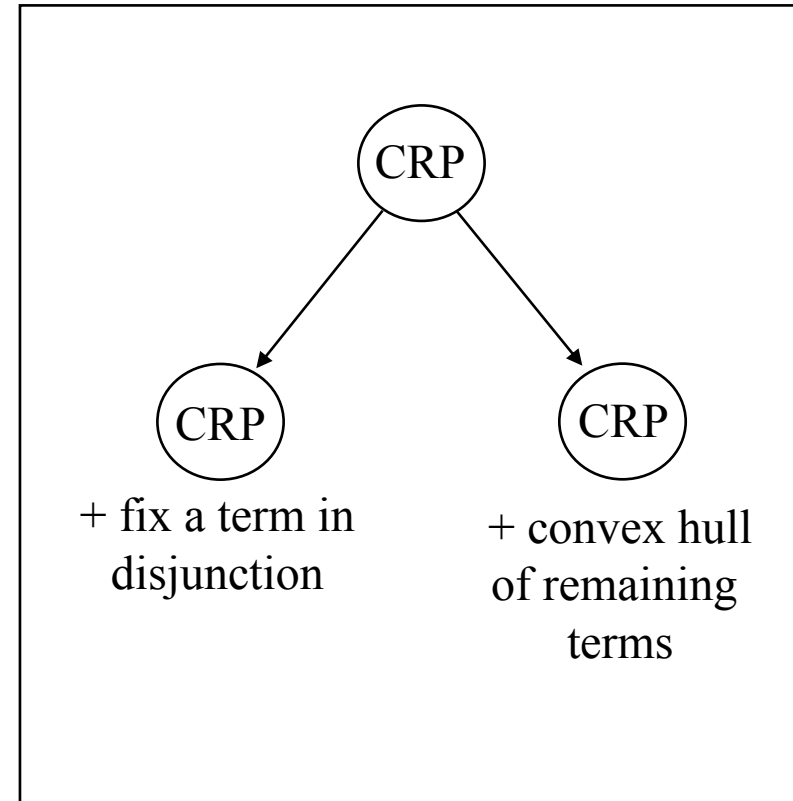
Proposition: The projected relaxation of (CH) onto the space of (BM) is always as tight or tighter than that of (BM) (Grossmann I.E. , S. Lee, 2003)

Trade-off: *Big-M fewer vars/weaker relaxation vs Convex-Hull tighter relaxation/more vars*



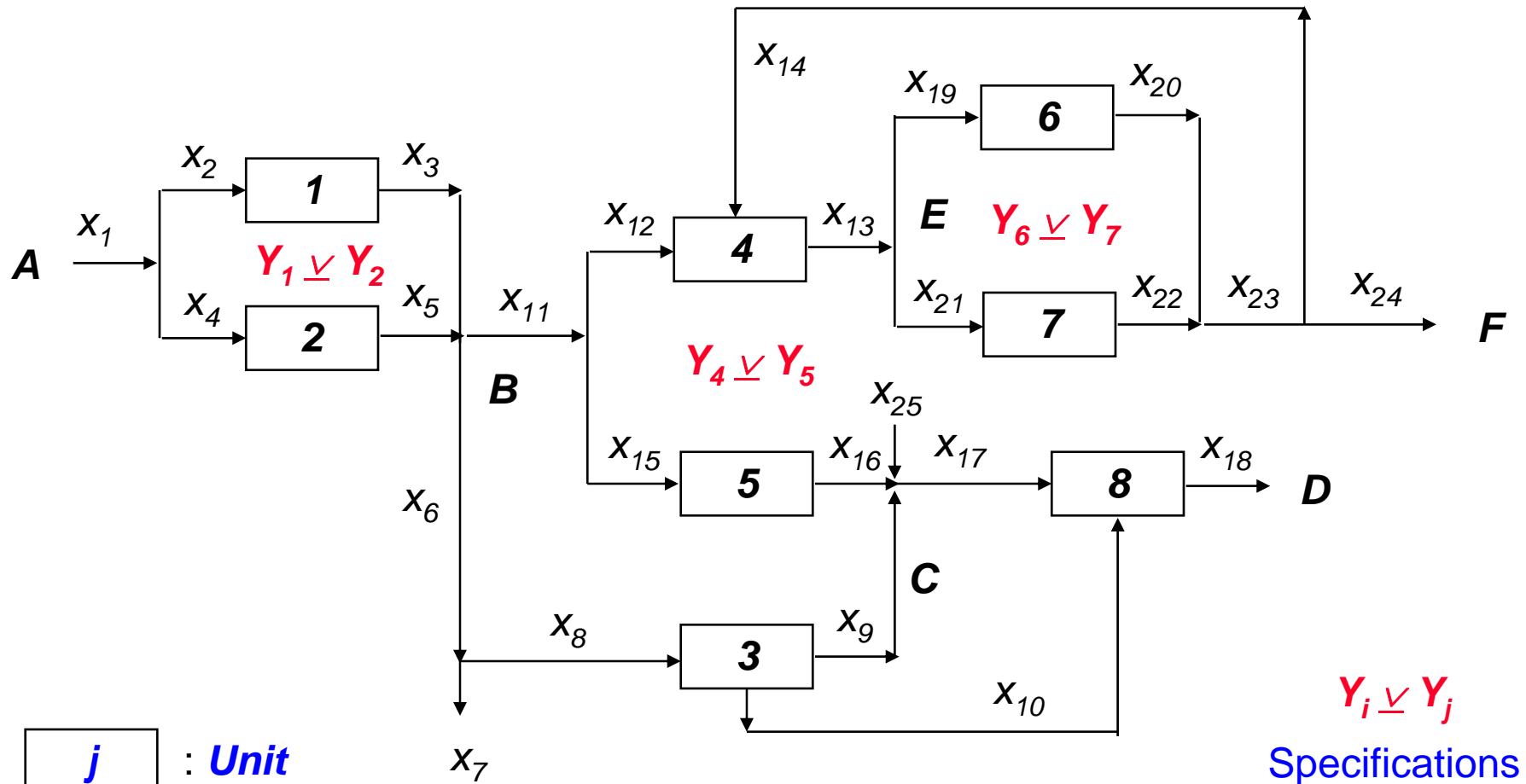


- **Tree Search**
 - ◆ NLP subproblem at each node
- **Solve Relaxation GDP (CRP)**
 - ◆ lower bound
- **Branching Rule**
 - ◆ Set the largest λ_j as 1
 - ◆ Dichotomy rule
- **Logic inference**
CNF unit resolution (Raman & Grossmann, 1993)
- **Depth first search**
When all the terms are fixed
upper bound
- Repeat Branching until $Z^L > Z^U$.



Process Network with Fixed Charges

- *Türkay and Grossmann (1997)*
 - ◆ Superstructure of the process

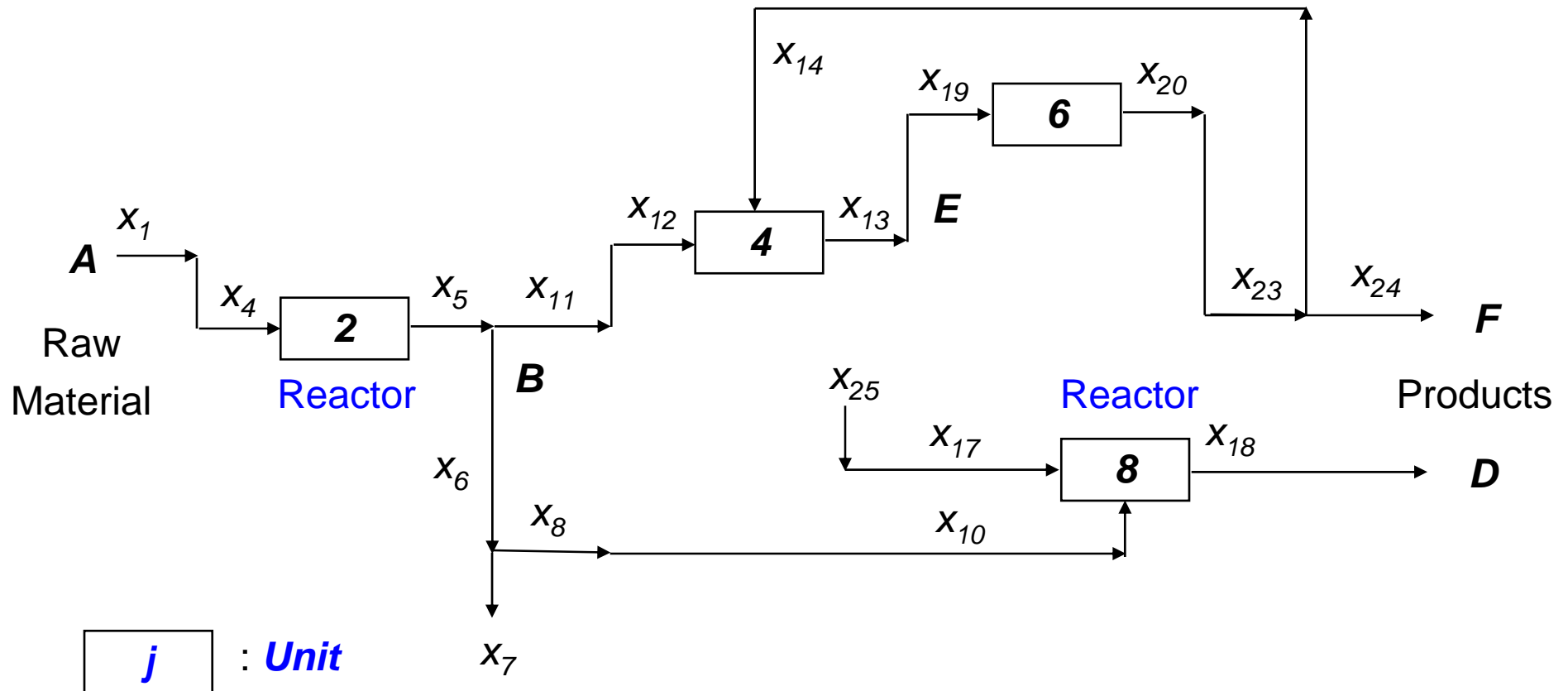


$Y_i \leq Y_j$
Specifications

j : Unit
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Optimal solution

- ◆ Minimum Cost: \$ 68.01M/year

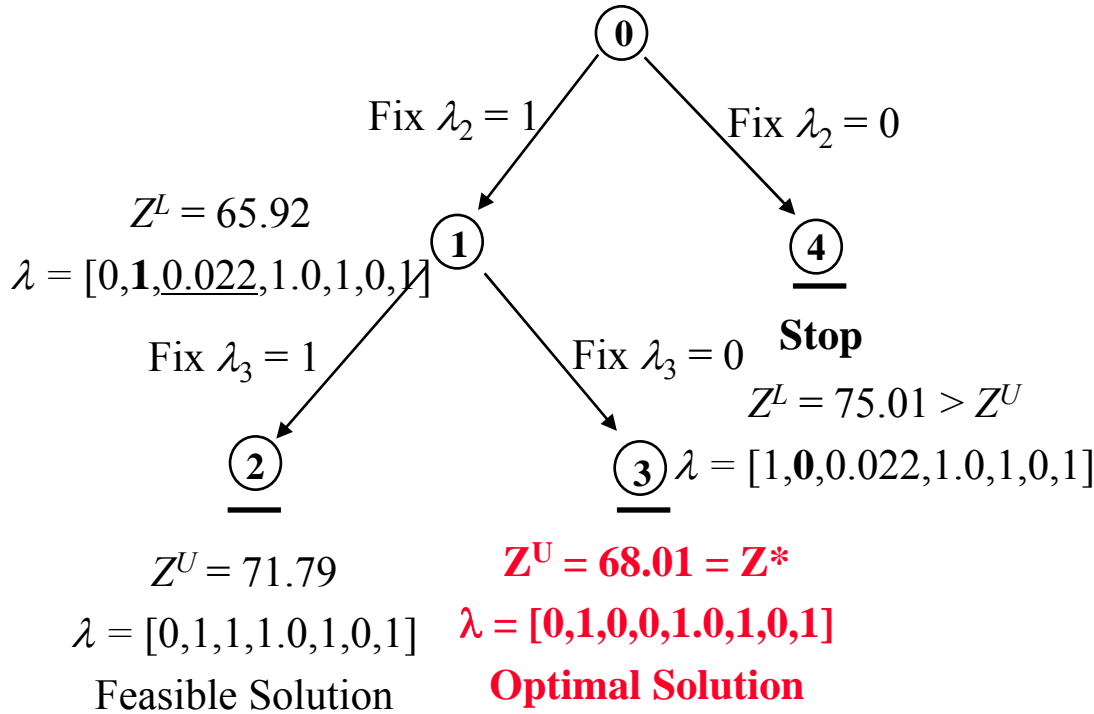


Proposed BB Method

Proposed BB

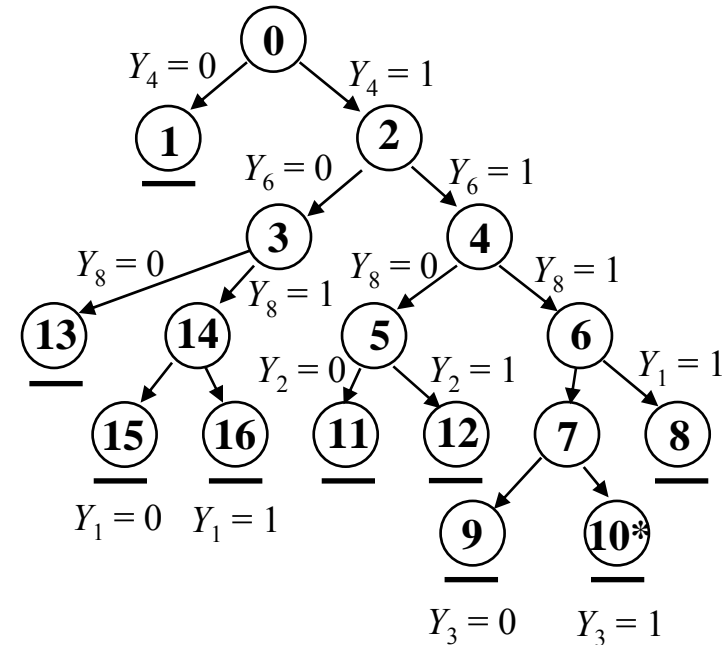
$Z^L = 62.48$

$$\lambda = [0.31, 0.69, 0.03, 1.0, 1.0, 1]$$



$Z^L = 15.08$

Big-M Std. BB



- ◆ 5 nodes vs. 17 nodes of Standard BB (lower bound = 15.08)

Logic-based Outer Approximation

Main point: avoids solving MINLP in full space

NLP Subproblem:
(reduced)

$$\begin{aligned} \min Z &= \sum_{k \in SD} c_k + f(x) \\ \text{s.t. } &g(x) \leq 0 \\ &\left. \begin{aligned} h_{ik}(x) &\leq 0 \\ c_k &= \gamma_{ik} \end{aligned} \right\} \text{for } Y_{ik} = \text{true } \hat{i} \in D_k, k \in SD & \text{(NLPD)} \\ &\left. \begin{aligned} B^i x &= 0 \\ c_k &= 0 \end{aligned} \right\} \text{for } Y_{ik} = \text{false } i \in D_k, i \neq \hat{i}, k \in SD \\ &x \in R^n, c_j \in R^m, \end{aligned}$$

Turkay, Grossmann (1997)

**Redundant constraints
are eliminated with false
values**

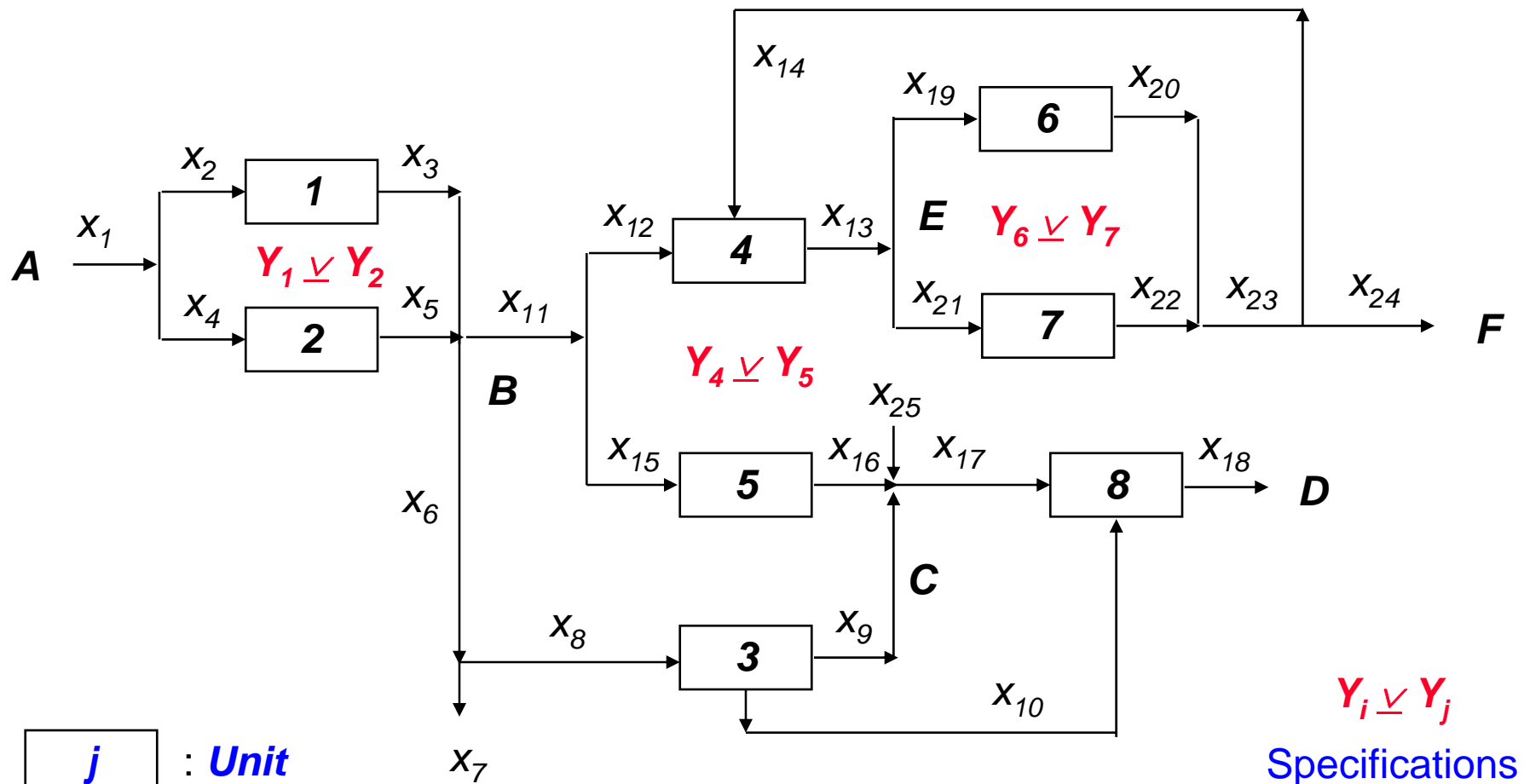
Master Problem:

$$\begin{aligned} \text{Min } Z &= \sum_k c_k + \alpha \\ \text{s.t. } &\left. \begin{aligned} \alpha &\geq f(x^l) + \nabla f(x^l)^T (x - x^l) \\ g(x^l) + \nabla g(x^l)^T (x - x^l) &\leq 0 \end{aligned} \right\} l = 1, \dots, L & \text{(MGDP)} \\ &\bigvee_{i \in D_k} \left[\begin{array}{l} Y_{ik} \\ h_{ik}(x^l) + \nabla h_{ik}(x^l)^T (x - x^l) \leq 0 \\ l \in L_{ik} \\ c_k = \gamma_{ik} \end{array} \right] & k \in SD \\ &\Omega(Y) = \text{True} \\ &\alpha \in R, x \in R^n, c \in R^m, Y \in \{\text{true}, \text{false}\}^m \end{aligned}$$

**Master problem solved with
disjunctive branch and bound or
with MILP reformulation**

Process Network with Fixed Charges

- *Türkay and Grossmann (1997)*
 - ◆ Superstructure of the process



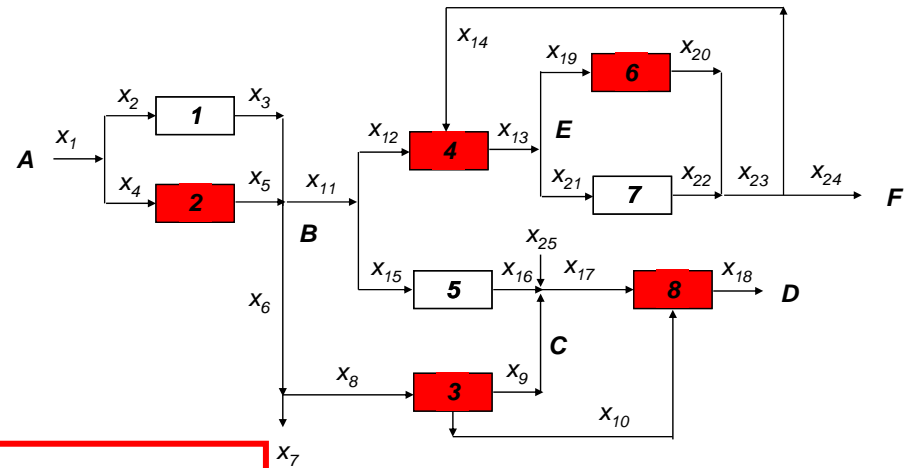
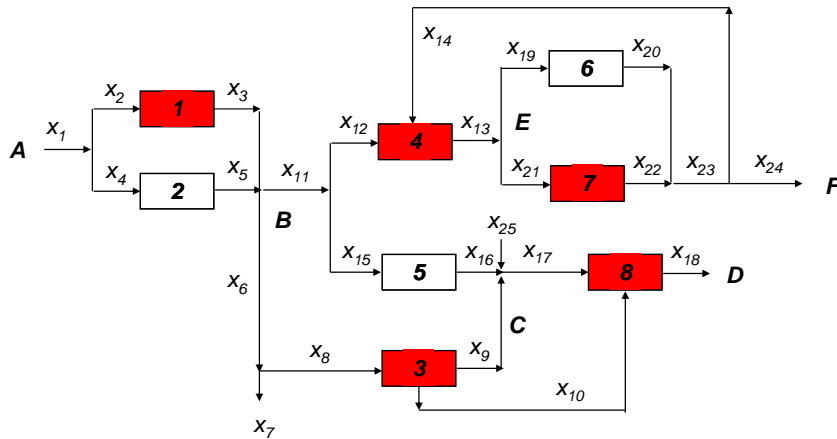
j : Unit
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$Y_i \leq Y_j$
Specifications

LOGMIP- Logic Based OA

NLP1 = 73.7

NLP2 = 103.6

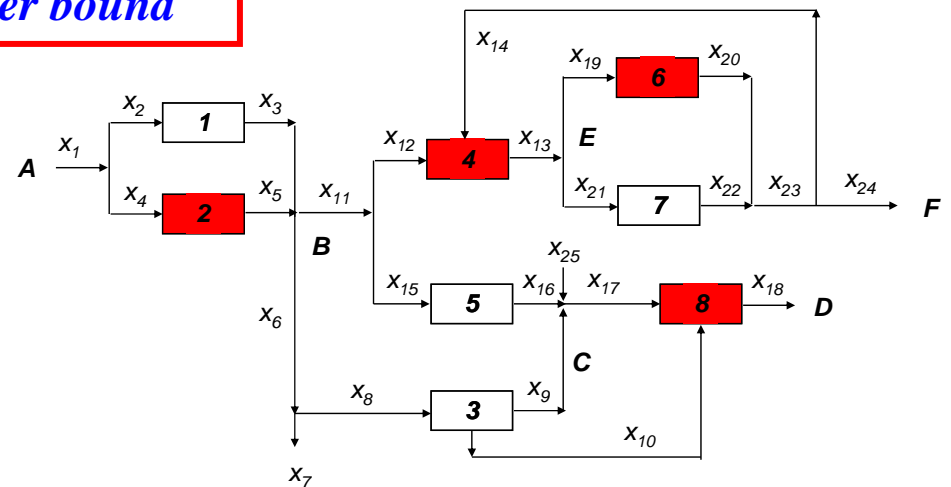
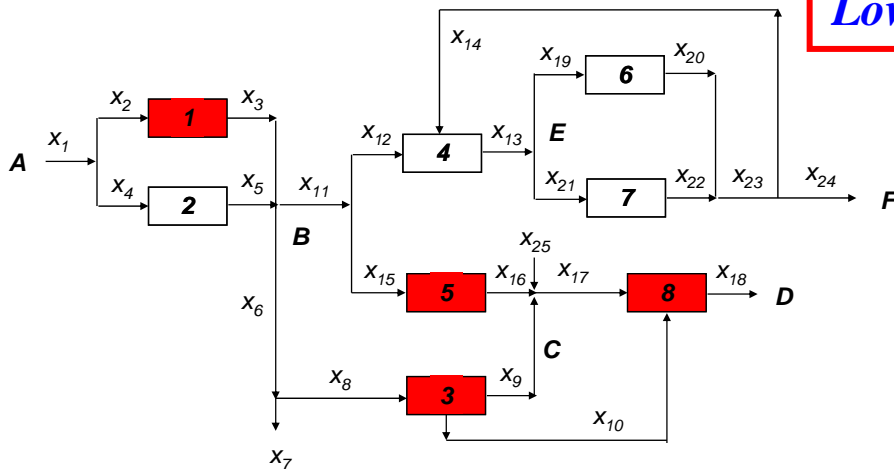


NLP3 = 133.8

MIP = 67.9

Lower bound

NLP4 = 68.0

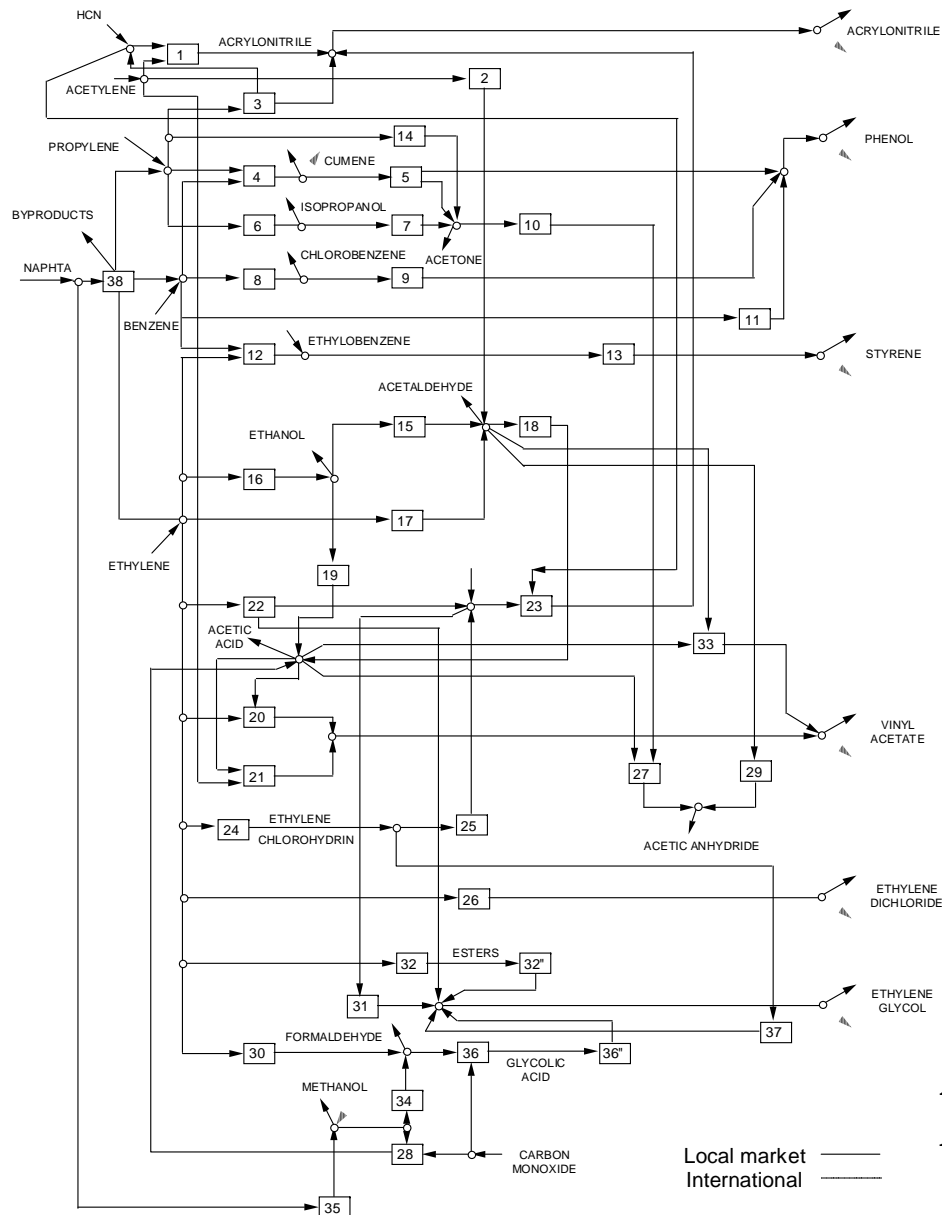


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Optimum!



Example: Optimal Production Planning with Contracts



LP Multiperiod Planning Model

38 processes

28 chemicals

10 months

Possibility contracts:

Naphtha, Ethylene, Acetylene

Purchases:

Fixed price

Discount after amount

Bulk discount after amount

Contracts fixed time

*Park, Park, Melle, Grossmann (2006)
I&EC Res., 45, 5013-5026*

Multiperiod Production Planning LP Model

Fixed price purchases

$$\begin{aligned} \text{Max } \text{PROFIT} = & \sum_{j \in J} \sum_{t \in T} \psi_{jt} S_{jt} - \sum_{j \in J} \sum_{t \in T} \phi_{jt} P_{jt} \\ & - \sum_{i \in I} \sum_{j \in JM_i} \sum_{t \in T} \delta_{it} W_{ijt} - \sum_{j \in J} \sum_{t \in T} \xi_{jt} V_{jt} - \sum_{j \in J} \sum_{t \in T} \theta_{jt} SF_{jt} \end{aligned}$$

Mass balance process

$$W_{ijt} = \mu_{ij} W_{ij't} \quad i \in I, j \in J_i, j' \in JM_i, t \in T$$

Capacity

$$W_{ijt} \leq Q_{it} \quad i \in I, j \in JM_i, t \in T$$

Mass balance chemicals

$$V_{j,t-1} + \sum_{i \in O_j} W_{ijt} + P_{jt} = V_{jt} + \sum_{i \in I_j} W_{ijt} + S_{jt} \quad j \in J, t \in T$$

Shortfalls

$$SF_{jt} \geq d_{jt}^U - S_{jt} \quad j \in J, t \in T$$

Purchases

$$\left. \begin{aligned} a_{jt}^L &\leq P_{jt} \leq a_{jt}^U \\ d_{jt}^L &\leq S_{jt} \leq d_{jt}^U \end{aligned} \right\} \quad j \in J, t \in T$$

Sales

**What if prices not fixed
but given by contracts?**

Limit inventory

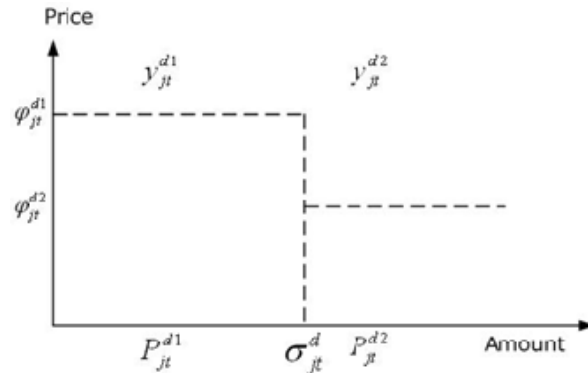
$$V_{jt} \leq V_{jt}^U \quad j \in J, t \in T$$

$$0 \leq SF_{jt} \leq SF_{jt}^U \quad j \in J, t \in T$$

$$S_{jt}, P_{jt}, W_{it}, V_{jt} \geq 0$$

Discount after σ_{jt}^d amount.

Price drops after amount σ



Disjunctive model

$$COST_{jt}^d = \varphi_{jt}^{d1} P_{jt}^{d1} + \varphi_{jt}^{d2} P_{jt}^{d2} \quad j \in JR, t \in T$$

$$\left[\begin{array}{l} y_{jt}^{d1} \\ 0 \leq P_{jt}^{d1} \leq \sigma_{jt}^d \\ P_{jt}^{d2} = 0 \end{array} \right] \vee \left[\begin{array}{l} y_{jt}^{d2} \\ P_{jt}^{d1} = \sigma_{jt}^d \\ P_{jt}^{d2} \geq 0 \end{array} \right] \quad j \in JR, t \in T$$

$$P_{jt}^d = P_{jt}^{d1} + P_{jt}^{d2} \quad j \in JR, t \in T$$

MILP model (Convex hull)

$$COST_{jt}^d = \varphi_{jt}^{d1} P_{jt}^{d1} + \varphi_{jt}^{d2} P_{jt}^{d2} \quad j \in JR, t \in T$$

$$P_{jt}^d = P_{jt}^{d1} + P_{jt}^{d2} \quad j \in JR, t \in T$$

$$P_{jt}^{d1} = P_{jt}^{d11} + P_{jt}^{d12} \quad j \in JR, t \in T$$

$$0 \leq P_{jt}^{d11} \leq y_{jt}^{d1} \sigma_{jt}^d \quad j \in JR, t \in T$$

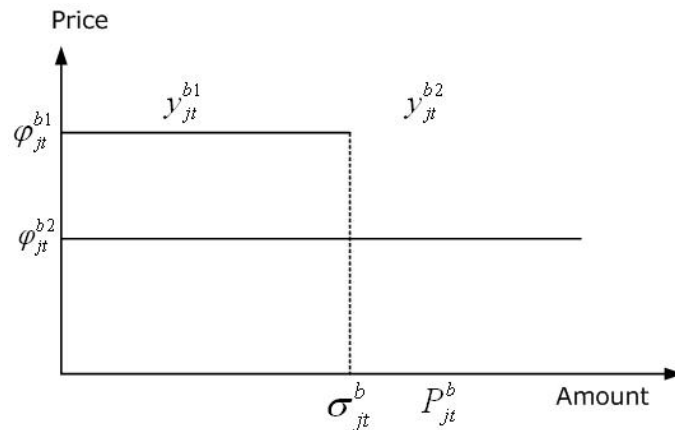
$$P_{jt}^{d12} = y_{jt}^{d2} \sigma_{jt}^d \quad j \in JR, t \in T$$

$$0 \leq P_{jt}^{d2} \leq y_{jt}^{d2} U_{jt}^d \quad j \in JR, t \in T$$

$$y_{jt}^{d1} + y_{jt}^{d2} = y_{jt}^d \quad j \in JR, t \in T$$

$$y_{jt}^{d1}, y_{jt}^{d2} \in \{0,1\}$$

Bulk discount



Disjunctive Model

$$\left[\begin{array}{l} y_{jt}^{b1} \\ COST_{jt}^b = \varphi_{jt}^{b1} P_{jt}^b \\ 0 \leq P_{jt}^b \leq \sigma_{jt}^b \end{array} \right] \vee \left[\begin{array}{l} y_{jt}^{b2} \\ COST_{jt}^b = \varphi_{jt}^{b2} P_{jt}^b \\ P_{jt}^b \geq \sigma_{jt}^b \end{array} \right] \quad j \in JR, t \in T$$

Price for total amount drops after amount σ

MILP Model (Convex Hull)

$$COST_{jt}^b = \varphi_{jt}^{b1} P_{jt}^{b1} + \varphi_{jt}^{b2} P_{jt}^{b2} \quad j \in JR, t \in T$$

$$P_{jt}^b = P_{jt}^{b1} + P_{jt}^{b2} \quad j \in JR, t \in T$$

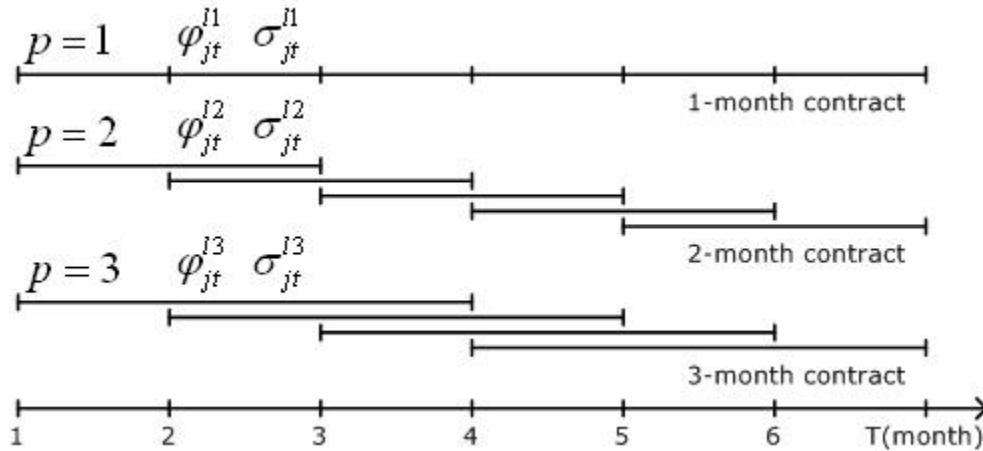
$$0 \leq P_{jt}^{b1} \leq \sigma_{jt}^b y_{jt}^{b1} \quad j \in JR, t \in T$$

$$\sigma_{jt}^b y_{jt}^{b2} \leq P_{jt}^{b2} \leq U_{jt}^b y_{jt}^{b2} \quad j \in JR, t \in T$$

$$y_{jt}^{b1} + y_{jt}^{b2} = y_{jt}^b \quad j \in JR, t \in T$$

$$y_{jt}^{b1}, y_{jt}^{b2} \in \{0,1\}$$

Fixed-duration contracts



Price gradually drops if amount σ purchased fixed number of months

Disjunctive Model

$$\left[\begin{array}{l} y_{jt}^l \\ COST_{jt}^l = \varphi_{jt}^l P_{jt}^l \\ P_{jt}^l \geq \sigma_{jt}^l \end{array} \right] \vee \left[\begin{array}{l} y_{jt}^{l2} \\ COST_{jt}^l = \varphi_{jt}^{l2} P_{jt}^l \\ COST_{j,t+1}^l = \varphi_{jt}^{l2} P_{j,t+1}^l \\ P_{jt}^l \geq \sigma_{jt}^{l2} \\ P_{j,t+1}^l \geq \sigma_{jt}^{l2} \end{array} \right] \vee \left[\begin{array}{l} y_{jt}^{l3} \\ COST_{jt}^l = \varphi_{jt}^{l3} P_{jt}^l \\ COST_{j,t+1}^l = \varphi_{jt}^{l3} P_{j,t+1}^l \\ COST_{j,t+2}^l = \varphi_{jt}^{l3} P_{j,t+2}^l \\ P_{jt}^l \geq \sigma_{jt}^{l3} \\ P_{j,t+1}^l \geq \sigma_{jt}^{l3} \\ P_{j,t+2}^l \geq \sigma_{jt}^{l3} \end{array} \right] \quad j \in JR, t \in T$$

MILP Model

$$COST_{jt}^l = \sum_{p \in LC} \sum_{\tau \in T_i^p} \varphi_{j\tau}^{lp} P_{j\tau}^{lp} \quad j \in JR, t \in T$$

$$P_{jt}^l = \sum_{p \in LC} \sum_{\tau \in T_i^p} P_{j\tau}^{lp} \quad j \in JR, t \in T$$

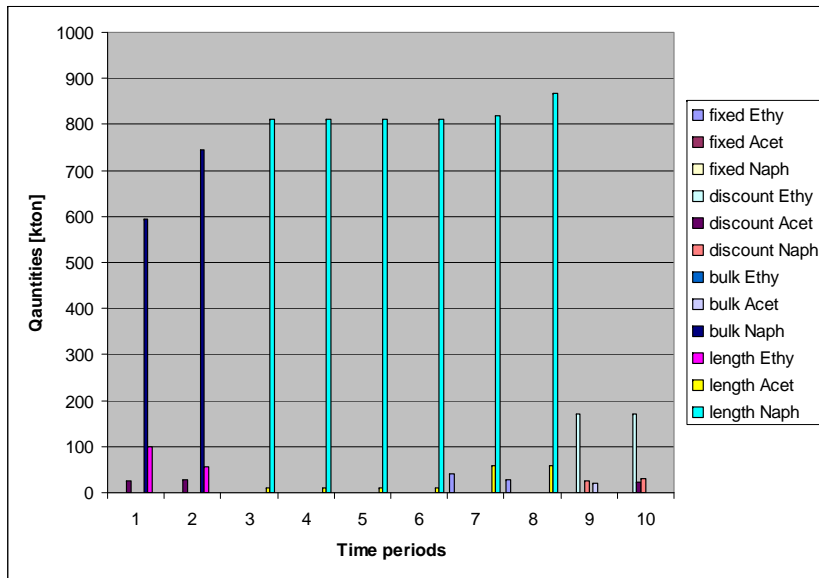
$$\sigma_{j\tau}^{lp} y_{j\tau}^{lp} \leq P_{j\tau}^{lp} \leq U_j^l y_{j\tau}^{lp} \quad j \in JR, t \in T_\tau^p \subset T, \tau \in T_i^p \subset T, p \in LC$$

$$\sum_{p \in LC} y_{j\tau}^{lp} \leq y_{j\tau}^l \quad j \in JR, \tau \in T$$

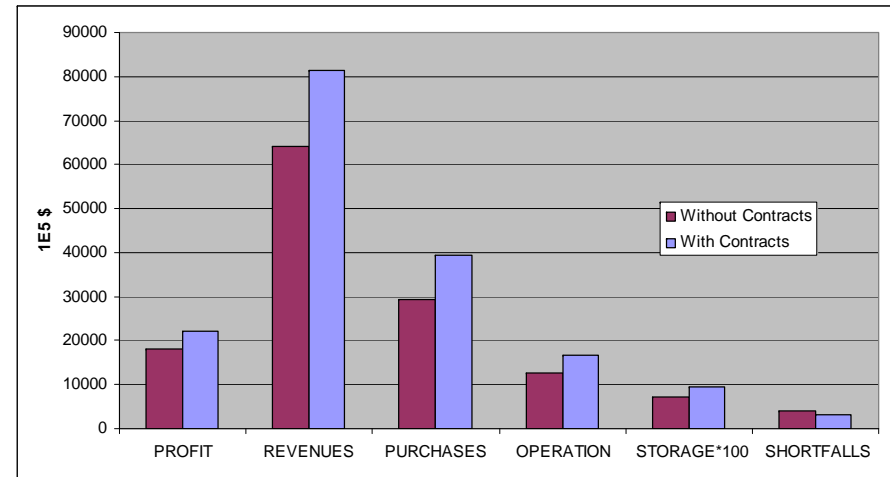
$$y_{j\tau}^{lp} \in \{0,1\}, LC = \{1,2,3\}$$

Computational Results

Case	0-1 variables	Cont. variables	Constraints	CPU time [s]	Time periods	Profit [10 ³ \$]
1	0	12,606	13,416	0.18	10	18,085.95
2	6,160	40,606	46,002	0.95	10	22,073.06



Purchases raw materials



Comparison without/with contracts



Computational Results



MODEL STATISTICS

BLOCKS OF EQUATIONS	47	SINGLE EQUATIONS	46,002
BLOCKS OF VARIABLES	30	SINGLE VARIABLES	40,606
NON ZERO ELEMENTS	85,033	DISCRETE VARIABLES	6,160

S O L V E S U M M A R Y

MODEL	MULT	OBJECTIVE	NPV
TYPE	MIP	DIRECTION	MAXIMIZE
SOLVER	CPLEX	FROM LINE	878

**** SOLVER STATUS 1 NORMAL COMPLETION
**** MODEL STATUS 1 OPTIMAL
**** OBJECTIVE VALUE 22073.0600

RESOURCE USAGE, LIMIT	0.470	10000.000
ITERATION COUNT, LIMIT	1437	1000000

ILOG CPLEX Dec 1, 2008 22.9.2 LNX 7311.8080 LX3 x86/Linux
Cplex 11.2.0, GAMS Link 34

MIP Presolve eliminated 44425 rows and 38571 columns.

Reduced MIP has 909 rows, 1367 columns, and 3267 nonzeros.

Reduced MIP has 270 binaries, 0 generals, 0 SOSs, and 0 indicators.

Implied bound cuts applied: 3

Flow cuts applied: 40

Gomory fractional cuts applied: 15

MIP Solution: 22073.060039 (959 iterations, 21 nodes)

Linear Generalized Disjunctive Programming LGDP Model

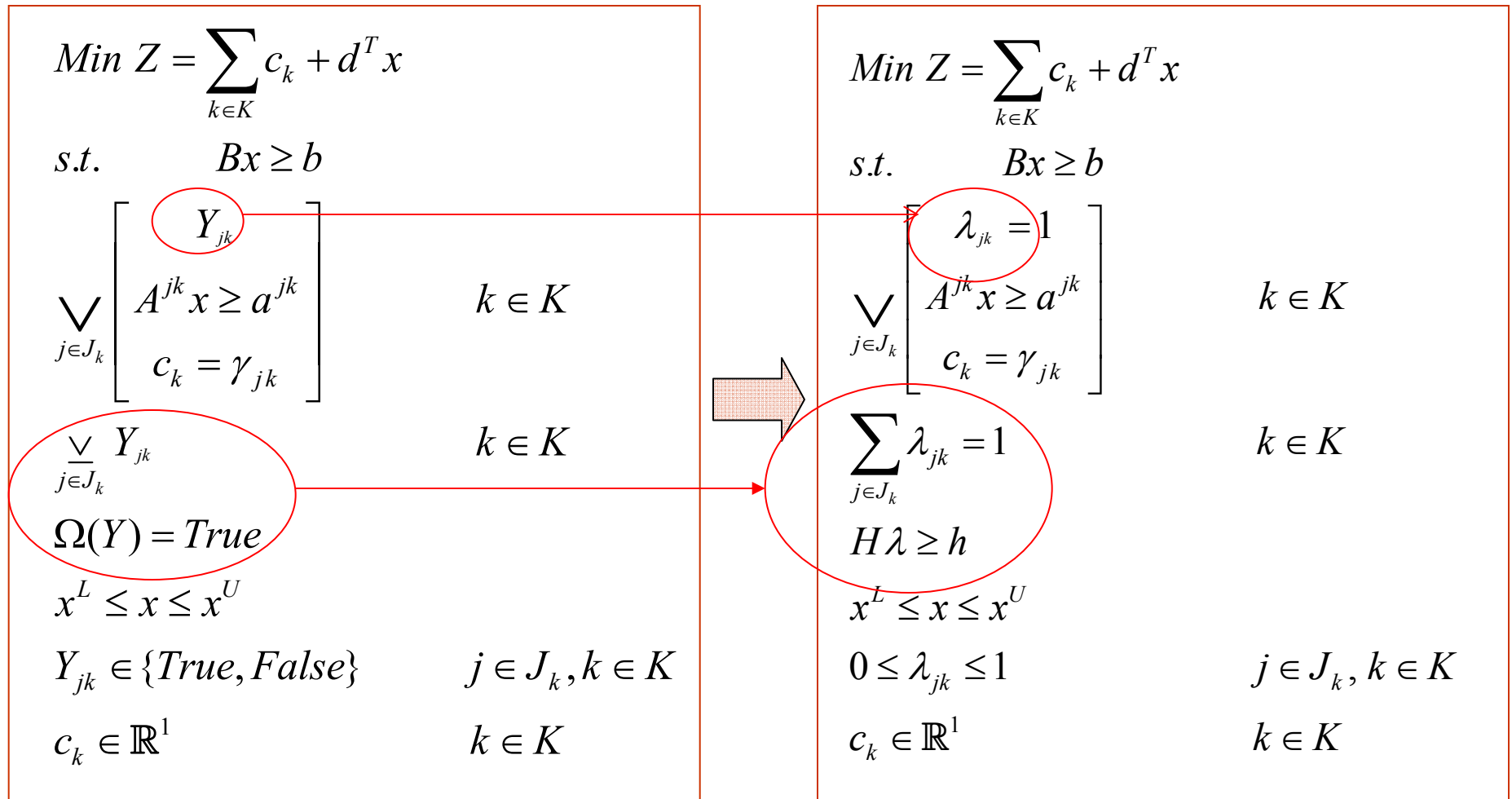
Sawaya N.W. and Grossmann I.E. (2008)

$\text{Min } Z = \sum_{k \in K} c_k + d^T x$	<i>Objective function</i>
$\text{s.t. } Bx \geq b$	<i>Common constraints</i>
$\bigvee_{j \in J_k} \left[\begin{array}{l} Y_{jk} \\ A^{jk} x \geq a^{jk} \\ c_k = \gamma_{jk} \end{array} \right]$	$k \in K$ ← <i>Disjunctive constraints</i>
$\bigvee_{j \in J_k} Y_{jk}$	$k \in K$
$\Omega(Y) = \text{True}$	← <i>Logic constraints</i>
$x^L \leq x \leq x^U$	← <i>Continuous variables</i>
$Y_{jk} \in \{\text{True}, \text{False}\}$	$j \in J_k, k \in K$ ← <i>Boolean variables</i>
$c_k \in \mathbb{R}^1$	$k \in K$

Logical OR operator

Can we obtain stronger relaxations?

Reformulating LGDP into Disjunctive Programming Formulation



LGDP

LDP => Integrality λ guaranteed

Proposition. LGDP and LDP have equivalent solutions.

Equivalent Forms in DP Through Basic Steps

There are many forms between CNF and DNF that are equivalent

Regular Form (RF): form represented by intersection of unions of polyhedra

Thus the RF is:

$$F = \bigcap_{t \in T} S_t$$

where for $t \in T$, $S_t = \bigcup_{i \in Q_t} P_i$, P_i a polyhedron, $i \in Q_t$.

Proposition 1 (Theorem 2.1 in Balas (1979)). *Let F be a disjunctive set in RF. Then F can be brought to DNF by $|T| - 1$ recursive applications of the following basic steps, which preserve regularity:*

For some $r, s \in T, r \neq s$, bring $S_r \cap S_s$ to DNF, by replacing it with:

$$S_{rs} = \bigcup_{\substack{i \in Q_r \\ t \in Q_s}} (P_i \cap P_t).$$

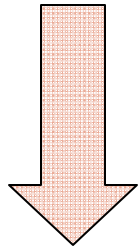
\Rightarrow as basic steps are performed tighter relaxations are obtained

Illustrative Example: Hierarchy of Relaxations

$$x_1 - x_2 + 0.5 \geq 0$$

$$-x_1 - x_2 + 1 \geq 0$$

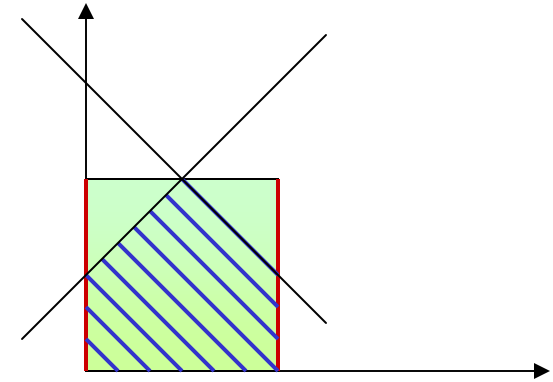
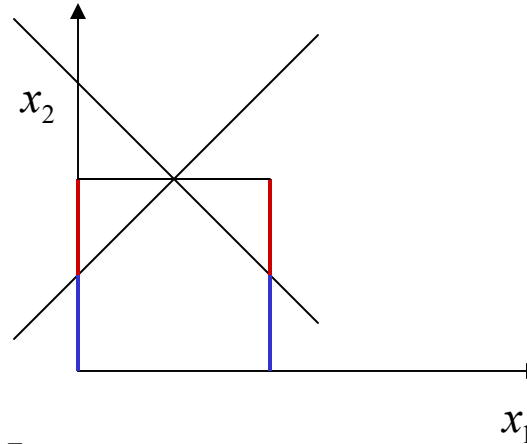
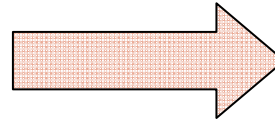
$$\left[\begin{array}{l} x_1 = 0 \\ 0 \leq x_2 \leq 1 \end{array} \right] \vee \left[\begin{array}{l} x_1 = 1 \\ 0 \leq x_2 \leq 1 \end{array} \right]$$



*Application of
2 Basic Steps*

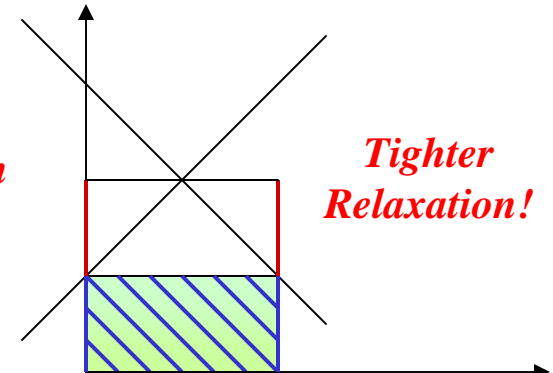
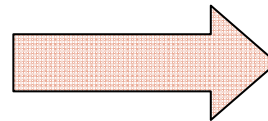
$$\left[\begin{array}{l} x_1 - x_2 + 0.5 \geq 0 \\ -x_1 - x_2 + 1 \geq 0 \\ x_1 = 0 \\ 0 \leq x_2 \leq 1 \end{array} \right] \vee \left[\begin{array}{l} x_1 - x_2 + 0.5 \geq 0 \\ -x_1 - x_2 + 1 \geq 0 \\ x_1 = 1 \\ 0 \leq x_2 \leq 1 \end{array} \right]$$

Convex Hull of disjunction



LP Relaxation

Convex Hull of disjunction



*Tighter
Relaxation!*

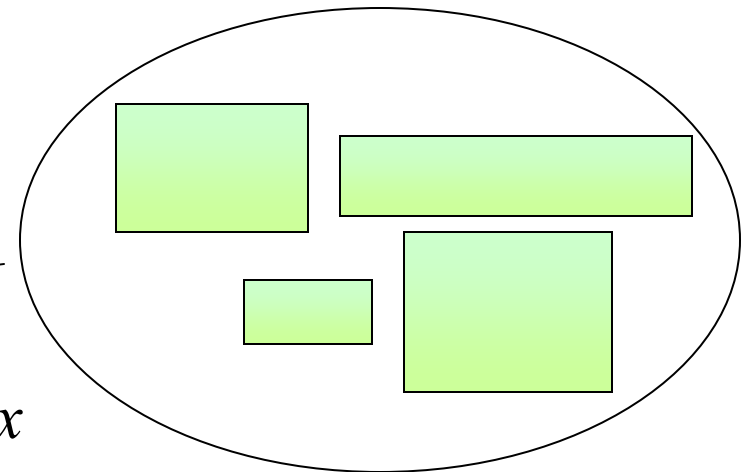
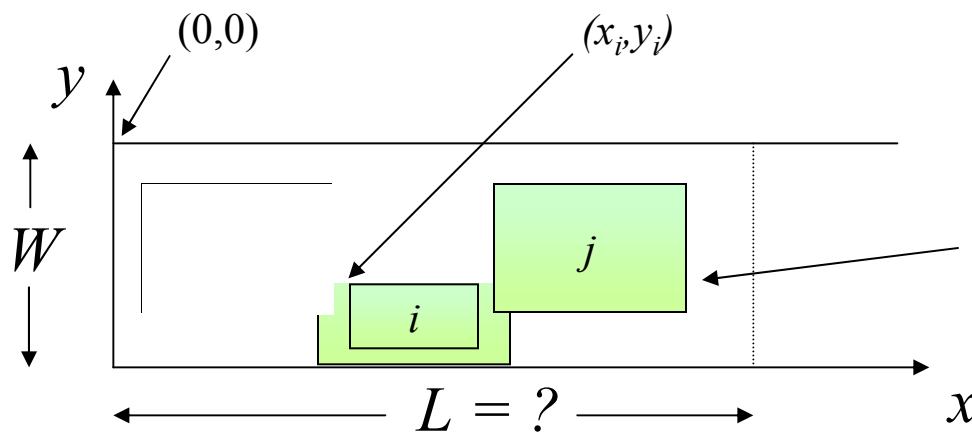
Numerical Example: Strip-packing problem

Problem statement: *Hifi (1998)*

Given a set of small rectangles with width H_i and length L_i .

Large rectangular strip of fixed width W and unknown length L .

Objective is to fit small rectangles onto strip without overlap and rotation while **minimizing length L of the strip.**



Set of small rectangles

GDP/DP Model for Strip-packing problem

<i>Min</i>	lt	
<i>s.t.</i>	$lt \geq x_i + L_i$	$\forall i \in N$
$\left[\begin{array}{c} Y_{ij}^1 \\ x_i + L_i \leq x_j \end{array} \right]$	$\vee \left[\begin{array}{c} Y_{ij}^2 \\ x_j + L_j \leq x_i \end{array} \right]$	$\vee \left[\begin{array}{c} Y_{ij}^3 \\ y_i - H_i \geq y_j \end{array} \right]$
	$\vee \left[\begin{array}{c} Y_{ij}^4 \\ y_j - H_j \geq y_i \end{array} \right]$	$\forall i, j \in N, i < j$
$x_i \leq UB_i - L_i$		$\forall i \in N$
$H_i \leq y_i \leq W$		$\forall i \in N$
$lt, x_i, y_i \in \mathbb{R}_+^1, Y_{ij}^1, Y_{ij}^2, Y_{ij}^3, Y_{ij}^4 \in \{True, False\}$		$\forall i, j \in N, i < j$

Objective function

Minimize length

Disjunctive constraints

No overlap between rectangles

Bounds on variables

25 Rectangle Problem Optimal solution= 31

Original CH

1,112 0-1 variables

4,940 cont vars

7,526 constraints

LP relaxation = 9

=>

Strengthened

1,112 0-1 variables

5,783 cont vars

8,232 constraints

LP relaxation = 27!

31 Rectangle Problem Optimal solution= 38

Original CH

2,256 0-1 variables

9,716 cont vars

14,911 constraints

LP relaxation = 10.64

=>

Strengthened

2,256 0-1 variables

11,452 cont vars

15,624 constraints

LP relaxation = 33!

Nonconvex GDP

$$\min Z = \sum_k c_k + f(x)$$

$$s.t. \quad r(x) \leq 0$$

OR operator \longrightarrow

$$\bigvee_{j \in J_k} \begin{bmatrix} Y_{jk} \\ g_{jk}(x) \leq 0 \\ c_k = \gamma_{jk} \end{bmatrix} \quad k \in K$$

$$\Omega(Y) = true$$

$$x \in R^n, c_k \in R^1$$

$$Y_{jk} \in \{ true, false \}$$

Objective Function

Common Constraints

Disjunctions

Logic Propositions

f, g and r: nonconvex

Convex Underestimator GDP (R)

- Introducing convex underestimators

$$\begin{aligned}
 \min Z &= \sum_k c_k + \bar{f}(x) \\
 \text{s.t.} \quad &\bar{r}(x) \leq 0 \\
 \bigvee_{j \in J_k} &\left[\begin{array}{l} Y_{jk} \\ \bar{g}_{jk}(x) \leq 0 \\ c_k = \gamma_{jk} \end{array} \right] \quad k \in K \\
 &\Omega(Y) = \text{true} \\
 &x \in R^n, c_k \in R^1 \\
 &Y_{jk} \in \{ \text{true}, \text{false} \} \\
 &\bar{f}, \bar{r} \text{ and } \bar{g} : \text{convex}
 \end{aligned}$$

Convex underestimators

Bilinear: Linear

McCormick (1976), Al-Khayyal (1992)

Linear fractional: Convex nonlinear

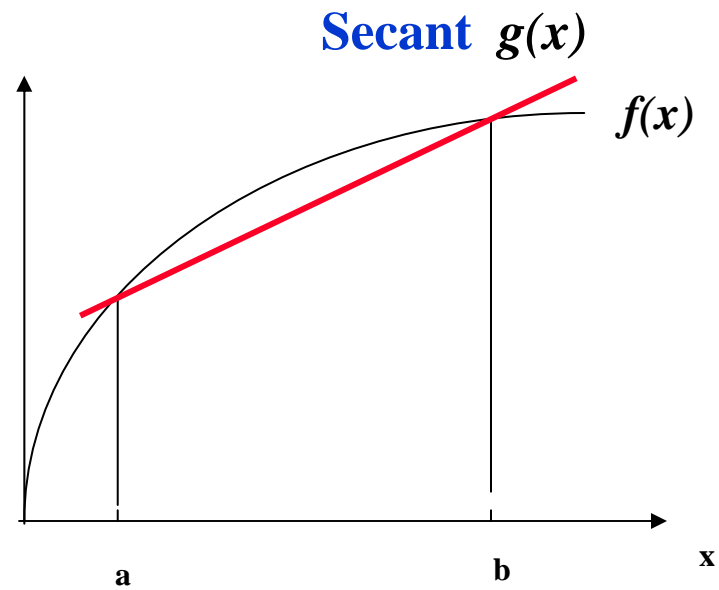
Quesada and Grossmann (1995)

Concave separable: Linear secant

- Problem (R) yields a valid lower bound to Problem (GDP)

Convex envelopes

Concave function



$$g(x) = f(a) + \frac{[f(b) - f(a)]}{b - a}(x - a)$$

Bilinear

$$w = xy$$

$$x^L \leq x \leq x^U \quad y^L \leq y \leq y^U$$

McCormick convex envelopes

$$w \geq x^L y + y^L x - x^L y^L$$

$$w \geq x^U y + y^U x - x^U y^U$$

$$w \leq x^L y + y^U x - x^L y^U$$

$$w \leq x^U y + y^L x - x^U y^L$$

For other convex envelopes/underestimators see:

Tawarmalani, M. and N. V. Sahinidis, *Convexification and Global Optimization in Continuous and Mixed-Integer Nonlinear Programming: Theory, Algorithms, Software, and Applications*, Vol. 65, *Nonconvex Optimization And Its Applications* series, Kluwer Academic Publishers, Dordrecht, 2002



Proposed framework to obtain stronger relaxations for nonconvex GDP

(Bilinear and Concave GDP)



(Ruiz & Grossmann, 2008)

The **framework** consists of **two main phases**:

- 1- Generate a **valid Linear Generalized Disjunctive Program (LGDP)** relaxation for the nonconvex GDP problem (e.g. bilinear and concave).
- 2- Strengthen the *continuous relaxation* of the **LGDP** obtained in phase 1 by **applying a set of basic steps**

Relaxation Results

	Global Optimum	Lower Bound (Lee & Grossmann Relaxation)	Lower Bound (Proposed Relaxation)	DNF Lower Bound
<i>Example 1</i>	-1.01	-1.28	-1.10	-1.10
<i>Example 2</i>	6.31	5.65	6.08	6.08
<i>Example 3</i>	114384.78	91671.18	94925.77	97858.86
<i>Example 4</i>	1214.87	400.66	431.90	431.90
<i>Example 5</i>	-4640	-5515	-5468	-5241

Remarks

-Proposed methodology leads to improvements in the lower bounds.

-Often, it is not necessary to reach the DNF form to have good lower bounds. Note that examples 1, 2 and 4 show that the lower bound is the same as the lower bound of the DNF

-The lower bound of the DNF is the best lower bound attainable for a given LGDP.

Conclusions

GDP modeling framework

- Provides a logic-based framework for *linear and nonlinear* discrete-continuous optimization
- big-M and convex hull alternative formulations *different relaxations*
- Solution methods: *reformulation, branch and bound, decomposition*
- Numerical example of *planning problem with contracts* has shown the impact of the convex hull formulation (*with help of CPLEX*)

Unified Linear GDP with Disjunctive Programming

- Developed *DP equivalent* formulation for GDP
- Numerical results have shown *great improvement in lower bound for strip packing problem*

Nonconvex GDPs

- *Tighter lower bounds* can be obtained in bilinear and concave problems by applying basic steps