MINLP and MPCC Strategies for Optimization of a Class of Hybrid Dynamic Systems

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May 21, 2009

Problem Statement

We first consider a switched system with ordinary differential equation models. This system has discontinuous right hand sides and is described by:

$$\varsigma(x(t), u(t), t) > 0, \dot{x} = f_+(x, u, t)$$
$$\varsigma(x(t), u(t), t) < 0, \dot{x} = f_-(x, u, t)$$
$$\varsigma(x(t), u(t), t) = 0, \dot{x} = \nu(t) f_-(x, u) + (1 - \nu(t)) f_+(x, u, t),\ \nu(t) \in [0, 1]$$

where $t, x(t), u(t), \nu(t)$ and $\varsigma(x(t), u(t), t))$ are the differential state variables, control variables, switching profiles, and switching (or guard) functions, respectively. The switching functions, ς determine transitions to different state models, represented by a switching profile with $\nu(t)$ set to zero or one; at the transition point, a convex combination of the two models is allowed. We extend to multiple index-1 DAE models, indexed by $m \in \mathcal{M}$ and switches that define N epochs of finite time, indexed by $i = 1, \ldots N$, given by:

$$\left.\begin{array}{l} F^m(\dot{x}, x(t), y(t), u(t), \nu_i(t), p) = 0 \\ g^m(x(t), y(t), u(t), p) \leq 0 \end{array}\right\} t \in (t_{i-1}, t_i], i = 1, \ldots N, m \in \mathcal{M}$$

with algebraic variables $y(t)$, time independent optimization variables p and inequality constraints g^m that include the switching functions. We further assume that epochs can be of variable time and that transitions occur only at epoch boundaries and that differential states remain continuous. This leads to a set of disjunctions represented through a relation between switching functions and corresponding switching profiles $w^m(x(t), \nu_i(t)) = 0$, Boolean variables Y_m and logical conditions $H(Y) = True$, represented by:

$$\bigvee_{m \in \mathcal{M}} \left[\begin{array}{l} \left. \begin{array}{l} F^m(\dot{x}, x(t), y(t), u(t), \nu_i(t), p) = 0 \\ g^m(x(t), y(t), u(t), p) \le 0 \\ w^m(x(t), \nu_i(t)) = 0 \end{array} \right\} t \in (t_{i-1}, t_i] Y_m \\ x(t_i^-) = x(t_i^+) \end{array} \right]$$

$$H(Y) = True,\ i = 1, \ldots, N$$

Our solution strategy for hybrid problems is to replace DAEs and their profiles by implicit Runge-Kutta (IRK), order 2K-1 approximations (e.g., $u(t) = \sum_{j=1}^{K} \psi_j(t) u_{ij}, t \in (t_i, t_{i+1}]$), with coefficients u_{ij} and $\psi_j(t)$ polynomial basis functions over finite elements of variable length h_i. (Because one or more finite elements are allowed to make up an epoch, we now adopt index i for finite elements.) We further represent disjunctions as complementarities at t_i and reformulate the resulting system as an MPCC solved with an NLP reformulation. Referring back to the ODE system, we can rewrite this system as:

$$\begin{aligned} \dot{x}(t_{ij}) &= \nu_{ij} f_-(x_{ij}, u_{ij}) + (1 - \nu_{ij}) f_+(x_{ij}, u_{ij}),\ \nu_{ij} \in [0, 1] \\ \varsigma(x_{ij}, u_{ij}) &= s_{ij}^+ - s_{ij}^-, \qquad i = 1, \ldots, N, j = 1, \ldots, K \\ 0 \le \sum_{k=0}^{K} s_{ik}^+ &\perp \nu_{ik'} \ge 0, \qquad i = 1, \ldots, N, k' = 1, \ldots, K \\ 0 \le \sum_{k=0}^{K} s_{ik}^- &\perp (1 - \nu_{ik'}) \ge 0, \qquad i = 1, \ldots, N, k' = 1, \ldots, K \end{aligned}$$

Note that the complementarities are formulated through slack variables that enforce either model over the entire epoch. In other words, $0 < \nu(t) < 1$ only if $\varsigma(x(t), u(t), t) = 0$ for the entire epoch.

To illustrate the approach, we consider a single differential equation with a switch that is covered by our class of hybrid systems. The example problem minimizes the objective function

$$\phi = (x_{end} - 5/3)^2 + \int_{t_0}^{t_{end}} x^2 \cdot dt$$

subject to a differential inclusion:

$$\dot{x} \in 2 - sgn(x), x(t_0) = -2.$$