Problem statement

Consider three processes $p_i(s) = \frac{b_i(s)}{a_i(s)}$, for $i = 1, 2, 3$. The problem of simultaneous stabilization of three systems (S3P) is to design a rational controller $c(s) = \frac{y(s)}{x(s)}$ which stabilizes all three processes at the same time. This latter condition is equivalent to requiring $a_i x + b_i y$ to be a Hurwitz stable polynomial for all i. A polynomial is stable if and only if all its roots lie in the open left half-plane. The problem of simultaneous *bistable* stabilization of one system (SB1P) is a special case of S3P with $p_2(s) = 1/0$ (which means that the controller must be *inverstable*) and $p_3(s) = 0/1$ (which means that the controller must be stable). There exist equivalence relations between S3P and SB1P showing that they are equivalent $\mathcal{NP}$-hard problems.

The Belgian chocolate problem (BCP) is similar to SB1P with $p_1(s) = \frac{b(s)}{a(s)}$. The only difference is that, in the Belgian chocolate problem, the inverstable requirement of the controller in SB1P is weakened to a *minimum phase* requirement, *i.e.* a controller whose zeros are ∞ or lie in the left half of the complex plane. The original statement of BCP is: For what values of $\delta \in \mathbb{R}$ is the scalar linear system $\frac{s^2-1}{s^2-2\delta s+1}$ stabilizable by a controller that is both stable and minimum phase? It is known that the process is stabilizable if and only if $\delta < \bar{\delta}$ for some $0.974 < \bar{\delta} < 0.9999800002$ [1, 2].

To locate $\bar{\delta}$, we need to solve the following optimization problem:

$$\begin{array}{ll} \max & \delta \\ \text{s.t.} & \deg(y) \le \deg(x) \\ & x(s), y(s), (ax + by)(s) \in \mathcal{SP}. \end{array}$$

where $\mathcal{SP}$ is the set of all stable real polynomials. Solving this conceptual problem to global optimality will solve the BCP. However, this problem has an infinite number of variables and constraints. To make the problem more manageable, we will fix the degree of the polynomial $x(s)$ to n giving:

$$\begin{array}{ll} \max & \delta \\ \text{s.t.} & x(s) \in \mathcal{MP}^n \\ & y(s) \in \mathcal{P}^n \\ & x(s), y(s), (ax + by)(s) \in \mathcal{SP} \end{array}$$

where $\mathcal{MP}^n$ is the set of real monic polynomials of degree n and $\mathcal{P}^n$ is the set of real polynomials of degree no more than n. Any feasible solution to this problem provides $\hat{\delta}$ for which a stable and minimum phase controller

$y(s)/x(s)$ of degree n can be used to stabilize the process. Therefore, $\hat{\delta}$ provides a valid lower bound for $\bar{\delta}$. By solving the problem to global optimality or increasing n, we will be able to improve the lower bound $\hat{\delta}$ approaching the true $\bar{\delta}$.

References

[1] V. D. Blondel. *Simultaneous Stabilization of Linear Systems.* Springer-Verlag, London, 1994.

[2] Y. Chang and N. V. Sahinidis. Global optimization in stabilizing controller design. *Journal of Global Optimization.* **38**(4):509-526, 2007.